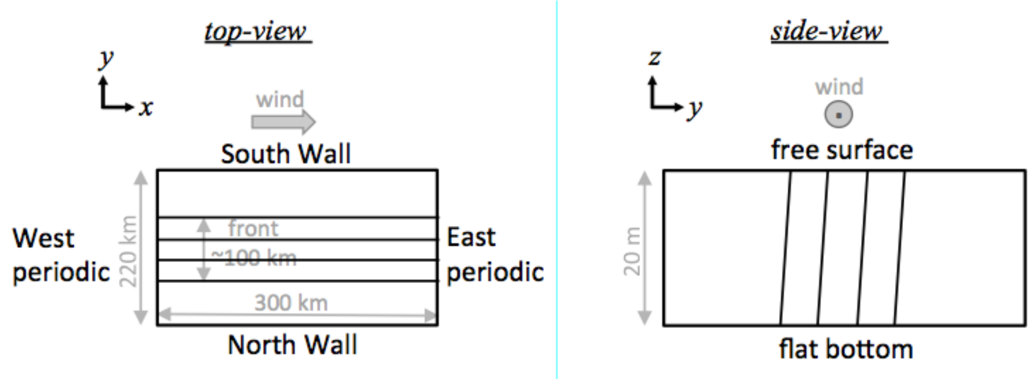
Goal: To calculate mean kinetic energy (MKE) and eddy kinetic energy (EKE) budgets from ROMS 3D momentum diagnostics
Model problem: Eady problem of baroclinic instability with surface wind forcing (see domain & forcing below)
Model domain: periodic in E-W (x-direction), N-S wall (y-direction), flat-bottom



Formulations:
MKE and EKE calculations follow von stroch (2012). The overbar below denotes zonal average (in x-direction). Prime denotes deviations from the zonal mean ($u' = u - \bar{u}$).

--- MKE bdget (x-component) = $\overline{\bar{u}\ (x\text{ - }mom\ \text{eq.})}$
--- EKE budget (x-component) = $\overline{u'\ (x\text{ - }mom\ \text{eq.})}$

$$\frac{\partial}{\partial t} MKE \left\{ \begin{aligned} \frac{\partial}{\partial t}\left(\frac{\bar{u}^2}{2}\right) &= \bar{u}\,\overline{u_{accel}} = \boxed{\bar{u}\,\overline{u_{xadv}} + \bar{u}\,\overline{u_{yadv}} + \bar{u}\,\overline{u_{vadv}}} + \bar{u}\,\overline{u_{cor}} + \bar{u}\,\overline{u_{prsgrd}} + \bar{u}\,\overline{u_{vvisc}} \\ \frac{\partial}{\partial t}\left(\frac{\bar{v}^2}{2}\right) &= \bar{v}\,\overline{v_{accel}} = \boxed{\bar{v}\,\overline{v_{xadv}} + \bar{v}\,\overline{v_{yadv}} + \bar{v}\,\overline{v_{vadv}}} + \bar{v}\,\overline{v_{cor}} + \bar{v}\,\overline{v_{prsgrd}} + \bar{v}\,\overline{v_{vvisc}} \end{aligned} \right.$$

$$\frac{\partial}{\partial t} EKE \left\{ \begin{aligned} \frac{\partial}{\partial t}\left(\frac{\overline{u'^2}}{2}\right) &= \overline{u'\,u_{accel}} = \boxed{\overline{u'\,u_{xadv}} + \overline{u'\,u_{yadv}} + \overline{u'\,u_{vadv}}} + \overline{u'\,u_{cor}} + \overline{u'\,u_{prsgrd}} + \overline{u'\,u_{vvisc}} \\ \frac{\partial}{\partial t}\left(\frac{\overline{v'^2}}{2}\right) &= \overline{v'\,v_{accel}} = \boxed{\overline{v'\,v_{xadv}} + \overline{v'\,v_{yadv}} + \overline{v'\,v_{vadv}}} + \overline{v'\,v_{cor}} + \overline{v'\,v_{prsgrd}} + \overline{v'\,v_{vvisc}} \end{aligned} \right.$$

Expectation:

After volume integration over a simple domain we use (E-W periodic, N-S closed), we expect (a) the "barotropic energy conversion" to come out from the advection terms, and (b) the barotropic conversion is of equal magnitude and opposite sign in MKE and EKE budgets (i.e. the conversion terms cancel each other when we add MKE and EKE). (see derivations below)

Problem: The conversion terms do not cancel each other! (see the figure below)

<u>Derivations: MKE</u>

from advection terms

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\bar{u}^2}{2}\right) = & -\frac{\partial}{\partial x}\left(\bar{u}\frac{\bar{u}^2}{2}\right) \boxed{-\frac{\partial}{\partial y}\left(\bar{v}\frac{\bar{u}^2}{2}\right)-\frac{\partial}{\partial z}\left(\bar{w}\frac{\bar{u}^2}{2}\right)} -\frac{\partial}{\partial x}\left(\bar{u}\,\overline{u'u'}\right) \boxed{-\frac{\partial}{\partial y}\left(\bar{u}\,\overline{v'u'}\right) \quad -\frac{\partial}{\partial z}\left(\bar{u}\,\overline{w'u'}\right)} + f\,\bar{u}\,\bar{v} \quad -\frac{1}{\rho_0}\frac{\partial}{\partial x}(\bar{u}\,\bar{p}) \quad +\frac{1}{\rho_0}\frac{\partial}{\partial z}(\bar{u}\,\bar{\tau}_x) \\
& +\frac{\bar{u}^2}{2}\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial x}+\frac{\partial \bar{w}}{\partial x}\right) \quad +\overline{u'u'}\frac{\partial \bar{u}}{\partial x} \quad +\overline{v'u'}\frac{\partial \bar{u}}{\partial y} \quad +\overline{w'u'}\frac{\partial \bar{u}}{\partial z} \quad +\frac{1}{\rho_0}\bar{p}\frac{\partial \bar{u}}{\partial x} \quad -\frac{1}{\rho_0}\bar{\tau}_x\frac{\partial \bar{u}}{\partial z} \\
& +\bar{u}\,\overline{u'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial x}+\frac{\partial w'}{\partial x}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\bar{v}^2}{2}\right) = & -\frac{\partial}{\partial x}\left(\bar{u}\frac{\bar{v}^2}{2}\right) \boxed{-\frac{\partial}{\partial y}\left(\bar{v}\frac{\bar{v}^2}{2}\right)-\frac{\partial}{\partial z}\left(\bar{w}\frac{\bar{v}^2}{2}\right)} -\frac{\partial}{\partial x}\left(\bar{v}\,\overline{u'v'}\right) \boxed{-\frac{\partial}{\partial y}\left(\bar{v}\,\overline{v'v'}\right) \quad -\frac{\partial}{\partial z}\left(\bar{v}\,\overline{w'v'}\right)} - f\,\bar{u}\,\bar{v} \quad -\frac{1}{\rho_0}\frac{\partial}{\partial y}(\bar{v}\,\bar{p}) \quad +\frac{1}{\rho_0}\frac{\partial}{\partial z}(\bar{v}\,\bar{\tau}_y) \\
& +\frac{\bar{v}^2}{2}\left(\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial x}+\frac{\partial \bar{w}}{\partial x}\right) \quad +\overline{u'v'}\frac{\partial \bar{v}}{\partial x} \quad +\overline{v'v'}\frac{\partial \bar{v}}{\partial y} \quad +\overline{w'v'}\frac{\partial \bar{v}}{\partial z} \quad +\frac{1}{\rho_0}\bar{p}\frac{\partial \bar{v}}{\partial y} \quad -\frac{1}{\rho_0}\bar{\tau}_y\frac{\partial \bar{v}}{\partial z} \\
& +\bar{v}\,\overline{v'\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial x}+\frac{\partial w'}{\partial x}\right)}
\end{aligned}
$$

$$
\begin{aligned}
0 = & -\frac{1}{\rho_0}\frac{\partial}{\partial z}(\bar{w}\,\bar{p}) \quad -\frac{g}{\rho_0}\bar{w}\,\bar{\rho} \\
& +\frac{1}{\rho_0}\bar{p}\frac{\partial \bar{w}}{\partial z}
\end{aligned}
$$

We can drop some terms <u>after volume integration</u> because

1. d/dx = 0 in a E-W periodic channel (green arrows)
2. Continuity equation for both mean and perturbations (red arrows)
3. At N-S wall boundaries, v = 0. If we do the analysis before perturbations reach N-S boundaries, then ($v = \bar{v} = 0$). After volume integration, the terms with yellow arrows drop off (??)
4. We are not so sure about the terms in the Orange box. We estimate the magnitude of these terms from ROMS history files (again, after volume integration). They are two-orders-of-magnitude smaller than others. So we assume that they are small.

Derivations: EKE

$$\frac{\partial}{\partial t}\overline{\left(\frac{u'^2}{2}\right)} = -\cancel{\frac{\partial}{\partial x}}\left(\bar{u}\,\overline{\frac{u'^2}{2}}\right) \boxed{-\frac{\partial}{\partial y}\left(\bar{v}\,\overline{\frac{u'^2}{2}}\right)-\frac{\partial}{\partial z}\left(\bar{w}\,\overline{\frac{u'^2}{2}}\right)} -\cancel{\frac{\partial}{\partial x}}\overline{\left(u'\frac{u'^2}{2}\right)} \boxed{-\frac{\partial}{\partial y}\overline{\left(v'\frac{u'^2}{2}\right)}-\frac{\partial}{\partial z}\overline{\left(w'\frac{u'^2}{2}\right)}} \cancel{+f\,\overline{u'v'}} -\frac{1}{\rho_0}\cancel{\frac{\partial}{\partial x}}\left(\overline{u'p'}\right) +\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\overline{u'\tau'_x}\right)$$

$$+\overline{\frac{u'^2}{2}}\cancel{\left(\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{w}}{\partial x}\right)} \quad -\overline{u'u'}\cancel{\frac{\partial\bar{u}}{\partial x}} \quad -\overline{v'u'}\frac{\partial\bar{u}}{\partial y} \quad -\overline{w'u'}\frac{\partial\bar{u}}{\partial z} \quad +\frac{1}{\rho_0}\overline{p'\cancel{\frac{\partial u'}{\partial x}}} \quad -\frac{1}{\rho_0}\overline{\tau'_x\frac{\partial u'}{\partial z}}$$

$$+\overline{\frac{u'^2}{2}\cancel{\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial x}+\frac{\partial w'}{\partial x}\right)}}$$

$$\frac{\partial}{\partial t}\overline{\left(\frac{v'^2}{2}\right)} = -\cancel{\frac{\partial}{\partial x}}\left(\bar{u}\,\overline{\frac{v'^2}{2}}\right) \boxed{-\frac{\partial}{\partial y}\left(\bar{v}\,\overline{\frac{v'^2}{2}}\right)-\frac{\partial}{\partial z}\left(\bar{w}\,\overline{\frac{v'^2}{2}}\right)} -\cancel{\frac{\partial}{\partial x}}\overline{\left(u'\frac{v'^2}{2}\right)} \boxed{-\frac{\partial}{\partial y}\overline{\left(v'\frac{v'^2}{2}\right)}-\frac{\partial}{\partial z}\overline{\left(w'\frac{v'^2}{2}\right)}} \cancel{-f\,\overline{u'v'}} -\frac{1}{\rho_0}\cancel{\frac{\partial}{\partial y}\left(\overline{v'p'}\right)} +\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\overline{v'\tau'_y}\right)$$

$$+\overline{\frac{v'^2}{2}}\cancel{\left(\frac{\partial\bar{u}}{\partial x}+\frac{\partial\bar{v}}{\partial x}+\frac{\partial\bar{w}}{\partial x}\right)} \quad -\overline{u'v'}\cancel{\frac{\partial\bar{v}}{\partial x}} \quad -\overline{v'v'}\frac{\partial\bar{v}}{\partial y} \quad -\overline{w'v'}\frac{\partial\bar{v}}{\partial z} \quad +\frac{1}{\rho_0}\overline{p'\cancel{\frac{\partial v'}{\partial y}}} \quad -\frac{1}{\rho_0}\overline{\tau'_y\frac{\partial v'}{\partial z}}$$

$$+\overline{\frac{v'^2}{2}\cancel{\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial x}+\frac{\partial w'}{\partial x}\right)}}$$

$$0 = -\frac{1}{\rho_0}\cancel{\frac{\partial}{\partial z}\left(\overline{w'p'}\right)} -\frac{g}{\rho_0}\overline{w'\rho'} +\frac{1}{\rho_0}\overline{p'\cancel{\frac{\partial w'}{\partial z}}}$$

Collecting the remaining terms, we obtain

$$\int \underline{\frac{\partial}{\partial t}\left(\frac{\bar{u}^2}{2}+\frac{\bar{v}^2}{2}\right)}\,dV = \int\left[\underbrace{\boxed{+\overline{v'u'}\frac{\partial\bar{u}}{\partial y}+\overline{w'u'}\frac{\partial\bar{u}}{\partial z}+\overline{v'v'}\frac{\partial\bar{v}}{\partial y}+\overline{w'v'}\frac{\partial\bar{v}}{\partial z}}}_{}^{\text{- C(Km, Ke)}} \underline{-\frac{g}{\rho_0}\bar{w}\bar{\rho}} \underline{+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\bar{u}\,\bar{\tau}_x\right)+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\bar{v}\,\bar{\tau}_y\right)-\frac{1}{\rho_0}\bar{\tau}_x\frac{\partial\bar{u}}{\partial z}-\frac{1}{\rho_0}\bar{\tau}_y\frac{\partial\bar{v}}{\partial z}}\right]dV$$

$$\int \underbrace{\frac{\partial}{\partial t}\overline{\left(\frac{u'^2}{2}+\frac{v'^2}{2}\right)}}_{\text{accel}}\,dV = \int\left[\underbrace{\boxed{-\overline{v'u'}\frac{\partial\bar{u}}{\partial y}-\overline{w'u'}\frac{\partial\bar{u}}{\partial z}-\overline{v'v'}\frac{\partial\bar{v}}{\partial y}-\overline{w'v'}\frac{\partial\bar{v}}{\partial z}}}_{\text{advection} \rightarrow \text{C(Km, Ke)}} \underbrace{-\frac{g}{\rho_0}\overline{w'\rho'}}_{\text{PGF+Cor}} \underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\overline{u'\tau'_x}\right)+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\overline{v'\tau'_y}\right)-\frac{1}{\rho_0}\overline{\tau'_x\frac{\partial u'}{\partial z}}-\frac{1}{\rho_0}\overline{\tau'_y\frac{\partial v'}{\partial z}}}_{\text{vvisc}}\right]dV$$

********** The main deductions here are ************

1. $C(K_m,K_e) = \int \overline{\bar{u}(\text{u\_xadv}+u\_\text{yadv}+u\_\text{vadv})}+\overline{\bar{v}(\text{v\_xadv}+v\_\text{yadv}+v\_\text{vadv})}dV$

2. $C(K_m,K_e)$ shows up in both MKE and EKE balance, but they have opposite sign. So, we expect

$$\int \overline{\bar{u}(\text{u\_xadv}+u\_\text{yadv}+u\_\text{vadv})}+\overline{\bar{v}(\text{v\_xadv}+v\_\text{yadv}+v\_\text{vadv})}dV \quad +$$

$$\int \overline{u'(\text{u\_xadv}+u\_\text{yadv}+u\_\text{vadv})}+\overline{v'(\text{v\_xadv}+v\_\text{yadv}+v\_\text{vadv})}dV \quad =0$$

Results —— The conversion terms ($C(K_m,K_e)$; blue curves) in MKE and EKE do not cancel each other !! We cannot figure out why. Are there errors in our calculations? Any suggestion is appreciated.

The result is shown in the figure below. The thick blue curves are from the advection terms (u*u_xadv+ …). We expect that the volume integration of advection terms is equal to the barotropic energy conversion. Note however that the blue curves in the EKE (top) and MKE (bottom) budgets are indeed of an opposite sign but they do not cancel each other. Advection in MKE ~ 3 x advection in EKE. BTW, we have checked the time variations of sigma level thickness to see if this contributes to the errors in KE tendency (e.g. Eq. 17 in MacCready and Giddings 2016). The thickness variation is quite small because the sea-level does not vary significantly with time.

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